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Existence, uniqueness and a priori estimates for a non linear integro - differential equation

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Abstract The paper deals with the explicit calculus and the properties of the fundamental solution K of a parabolic operator related to a semilinear equation that models reaction diffusion systems with excitable kinetics. The initial value problem in all of the space is analyzed together with continuous dependence and a priori estimates of the solution. These estimates show that the asymptotic behavior is determined by the reaction mechanism. Moreover it's possible a rigorous singular perturbation analysis for discussing travelling waves with their characteristic times.

Keywords Reaction - diffusion systems · Parabolic equations · Biological applications · Fundamental solutions · Laplace transform

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1 Statement of the problem and results

Let $\Omega_T \equiv \{ (x, t) : x \in \mathbb{R} \ 0 < t \leq T \}$ and, for functions $u(x, t) \in C^2(\Omega_T)$, let \mathcal{P}_0 be the non linear initial value problem defined in all of the space:

$$\begin{cases} Lu \equiv u_t - \varepsilon u_{xx} + au + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau = F(x, t, u) \\ u(x, 0) = g(x) \quad x \in \mathbb{R}, \end{cases} \quad (1)$$

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where a, b, ε, β are positive constants and F, g are known functions of their arguments.

If $K(x, t)$ is a fundamental solution of the parabolic operator L and $F(x, t, u)$ verifies appropriate assumptions, then the differential problem (1) is equivalent to the integral equation

$$u(x, t) = \mathcal{F}[u(x, t)], \quad (2)$$

where $\mathcal{F}[v]$ is the mapping

$$\begin{aligned} \mathcal{F}[v(x, t)] = & \int_{\mathbb{R}} K(x - \xi, t) g(\xi) d\xi + \\ & + \int_0^t d\tau \int_{\mathbb{R}} K(x - \xi, t - \tau) F[\xi, \tau, v(\xi, \tau)] d\xi. \end{aligned} \quad (3)$$

Now, let $\|v\|_T = \sup_{\Omega_T} |v(x, t)|$, and let \mathcal{B}_T denote the Banach space

$$\mathcal{B}_T \equiv \{v(x, t) : v \in C(\Omega_T), \|v\|_T < \infty\}. \quad (4)$$

The aim of the paper is the explicit calculus of K and the analysis of its basic properties (§2) useful to prove that \mathcal{F} is a contraction of \mathcal{B}_T in \mathcal{B}_T and hence admits a unique fixed point $u = u(x, t) \in \mathcal{B}_T$. Thus existence and uniqueness results for the problem \mathcal{P}_0 are deduced (§3), together with continuous dependence and a priori estimates of the solution (§4).

The following estimate (§2) is worthy of remark

$$|K(x, t)| \leq \frac{e^{-\frac{x^2}{4\varepsilon t}}}{2\sqrt{\pi\varepsilon t}} \left[e^{-at} + bt \frac{e^{-at} - e^{-\beta t}}{\beta - a} \right] \quad (5)$$

and it shows that K has the same basic properties of the fundamental solution of the heat operator and moreover decays exponentially to zero as t increases. As consequence (§4), the solution u of \mathcal{P}_0 is such that

$$\|u(x, t)\|_T \leq \|g\| (1 + \pi\sqrt{b}t) e^{-\omega t} + \beta_0 \|F\|, \quad (6)$$

where $\omega = \min(a, \beta)$, the norms are defined in §4 and the constant β_0 depends on a, b, β (see (30)). So, when t is large, the asymptotic behavior of u is determined by the properties of the source F .

The equation (1) models several engineering applications as motions of viscoelastic fluids or solids [3, 9, 12, 24]; heat conduction at low temperature [13, 20], sound propagation in viscous gases [16] and perturbed sine Gordon equation in the theory of the superconductivity [2, 7, 28] or in the propagation of localized magnetohydrodynamic models in plasma physics [30].

As an example, in section 5 the results are applied to the FitzHugh - Nagumo equations (FHN) which model many important biological phenomena [1, 11, 14, 15, 25]. According to well known results concerning systems of non linear reaction- diffusion equations [19, 31], estimates like (6) confirm that the large time behaviour of the solution is determined by the reaction mechanism. Moreover by means of the explicit fundamental solution K and its properties, it's possible a rigorous singular perturbation analysis to approximate travelling pulses in excitable media, together with their transition times.

2 Fundamental solution and properties

By denoting with

$$\hat{u}(x, s) = \int_0^\infty e^{-st} u(x, t) dt, \quad \hat{F}(x, s) = \int_0^\infty e^{-st} F[x, t, u(x, t)] dt,$$

the Laplace transform with respect to t , from (1) we get

$$\hat{u}(x, s) = \int_{\mathbb{R}} \hat{K}(x - \xi, s) [g(\xi) + \hat{F}(\xi, s)] d\xi, \quad (7)$$

where

$$\hat{K}(x, s) = \frac{e^{-\frac{|x|}{\sqrt{\varepsilon}} \sigma}}{2 \sqrt{\varepsilon} \sigma} \quad \text{with} \quad \sigma^2 = s + a + \frac{b}{s + \beta}. \quad (8)$$

Therefore, if $K(x, t)$ represents the inverse \mathcal{L}_t transforms of $\hat{K}(x, s)$, from (7) *formally* it follows that

$$\begin{aligned} u(x, t) &= \int_{\mathbb{R}} K(x - \xi, t) g(\xi) d\xi + \\ &+ \int_0^t d\tau \int_{\mathbb{R}} K(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi. \end{aligned} \quad (9)$$

If $r = |x|/\sqrt{\varepsilon}$ and $J_n(z)$ denotes the Bessel function of first kind and order n , let us consider the function

$$\begin{aligned} K(r, t) &= \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}} e^{-at} - \\ &- \frac{1}{2} \sqrt{\frac{b}{\pi\varepsilon}} \int_0^t \frac{e^{-\frac{r^2}{4y} - ay}}{\sqrt{t-y}} e^{-\beta(t-y)} J_1(2\sqrt{by(t-y)}) dy. \end{aligned} \quad (10)$$

Theorem 2.1- *In the half-plane $\Re s > \max(-a, -\beta)$ the Laplace integral $\mathcal{L}_t K(r, t)$ converges absolutely for all $r > 0$, and it results:*

$$\mathcal{L}_t K \equiv \int_0^\infty e^{-st} K(r, t) dt = \frac{e^{-r\sigma}}{2\sqrt{\varepsilon}\sigma}. \quad (11)$$

Proof - For all real z , one has $|J_n(z)| \leq 1$ and the Fubini -Tonelli theorem implies that

$$\begin{aligned} \mathcal{L}_t K &= \frac{e^{-r\sqrt{s+a}}}{2\sqrt{\varepsilon}(s+a)} - \\ &- \frac{\sqrt{b}}{2\sqrt{\pi\varepsilon}} \int_0^\infty e^{-(s+a)y - \frac{r^2}{4y}} dy \int_0^\infty e^{-(s+\beta)t} J_1(2\sqrt{by}t) \frac{dt}{\sqrt{t}}. \end{aligned}$$

Since

$$\int_0^\infty e^{-pt} \sqrt{\frac{c}{t}} J_1(2\sqrt{ct}) dt = 1 - e^{-c/p} \quad (\Re p > 0),$$

it follows that

$$\hat{K}(r, s) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_0^\infty e^{-\frac{r^2}{4y} - (s+a+\frac{b}{s+\beta})y} \frac{dy}{\sqrt{y}} = \frac{1}{2\sqrt{\varepsilon}} \frac{e^{-r\sigma}}{\sigma}. \quad \blacksquare$$

Theorem 2.2 - *The function K has the same basic properties of the fundamental solution of the heat equation, that is :*

A - $K(x, t) \in C^\infty$ for $t > 0$, $x \in \mathbb{R}$.

B - For fixed $t > 0$, K and its derivatives are vanishing esponentially fast as $|x|$ tends to infinity.

C - For any fixed $\delta > 0$, uniformly for all $|x| \geq \delta$, it results:

$$\lim_{t \downarrow 0} K(x, t) = 0, \quad (12)$$

D. - For $t > 0$, it is $LK = 0$.

Proof - By (10) the properties A-B-C are obvious. To verify D we put

$$\psi(r, t) = \frac{1}{2\sqrt{\pi\varepsilon t}} \exp \left[-\frac{r^2}{4t} - at \right] \quad (13)$$

$$\varphi(y, t) = \sqrt{\frac{by}{t-y}} J_1(2\sqrt{by(t-y)}) e^{-\beta(t-y)} \quad (14)$$

$$K(r, t) = \psi(r, t) - \int_0^t \psi(r, y) \varphi(y, t) dy. \quad (15)$$

As

$$(\partial_t + a - \partial_{rr}) \psi(r, t) = 0 \quad (16)$$

from (15) it results:

$$\begin{aligned} K_t + aK - K_{rr} &= -\varphi(t, t) \psi(r, t) - \int_0^t [\psi(\varphi_t + a\varphi) - \varphi\psi_{rr}] dy = \\ &= -\varphi(t, t) \psi(r, t) - \int_0^t [\psi(\varphi_t + a\varphi) - \varphi(\psi_y + a\psi)] dy. \end{aligned} \quad (17)$$

But

$$\int_0^t \varphi \psi_y dy = \varphi(t, t) \psi(r, t) - \int_0^t \psi \varphi_y dy$$

and so (17) gives

$$K_t + aK - K_{rr} = - \int_0^t \psi(\varphi_t + \varphi_y) dy. \quad (18)$$

As

$$(\partial_t + \partial_y) \varphi(y, t) = b J_0(2\sqrt{by(t-y)}) e^{-\beta(t-y)} \quad (19)$$

from (18) it follows

$$K_t + aK - K_{rr} = -b K_1 \quad (20)$$

where K_1 is given by

$$K_1(r, t) = \frac{1}{2\sqrt{\pi\varepsilon}} \int_0^t e^{-\frac{r^2}{4y} - ay - \beta(t-y)} J_0(2\sqrt{by(t-y)}) \frac{dy}{\sqrt{y}}. \quad (21)$$

On the other side, the convolution $e^{-\beta t} * K$ is

$$e^{-\beta t} * K = e^{-\beta t} * \psi - \int_0^t \psi(r, y) dy \int_y^t e^{-\beta(t-\tau)} \varphi(y, \tau) d\tau \quad (22)$$

and moreover it results:

$$\begin{aligned}
\int_y^t e^{-\beta(t-\tau)} \varphi(y, \tau) d\tau &= e^{-\beta(t-y)} \int_y^t \sqrt{\frac{by}{\tau-y}} J_1(2\sqrt{by(\tau-y)}) d\tau = \\
&= e^{-\beta(t-y)} \left[1 - J_0(2\sqrt{by(t-y)}) \right].
\end{aligned} \tag{23}$$

As consequence, from (22) - (23) we get

$$e^{-\beta t} * K = \int_0^t e^{-\beta(t-y)} \psi(r, y) J_0(2\sqrt{by(t-y)}) dy = K_1 \tag{24}$$

and so (20) - (24) imply property D because $K_{rr} = \varepsilon K_{xx}$. ■

Moreover, if $I_n(z)$ denotes the modified Bessel function of the real variable z and

$$E(t) = \frac{e^{-\beta t} - e^{-at}}{a - \beta} > 0, \tag{25}$$

then the following theorem holds:

Theorem 2.3 - *The fundamental solution $K(r, t)$ satisfies the estimates*

$$|K| \leq \frac{e^{-\frac{r^2}{4t}}}{2\sqrt{\pi\varepsilon t}} [e^{-at} + bt E(t)] \tag{26}$$

$$\int_{\mathbb{R}} |K(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{b} \pi t e^{-\frac{\beta+a}{2}t} I_0\left(\frac{\beta-a}{2}t\right). \tag{27}$$

Proof - As $|J_1(2\sqrt{by(t-y)})| \leq \sqrt{by(t-y)}$, from (10) we get (26). Further, because

$$\int_{\mathbb{R}} \frac{e^{-\frac{|x-\xi|^2}{4\varepsilon t}}}{2\sqrt{\pi\varepsilon t}} d\xi = 1,$$

from (10) and $|J_1(z)| \leq 1$ it results:

$$\begin{aligned}
\int_{\mathbb{R}} |K(x - \xi, t)| d\xi &\leq e^{-at} + \int_0^t e^{-ay - \beta(t-y)} \sqrt{\frac{by}{t-y}} dy = \\
&= e^{-at} + (\sqrt{b}/2) \pi t e^{-\frac{\beta+a}{2}t} \left[I_0\left(\frac{a-\beta}{2}t\right) - I_1\left(\frac{a-\beta}{2}t\right) \right].
\end{aligned}$$

As $I_1(-z) = -I_1(z)$, $I_0(-z) = I_0(z)$ and $I_1(|z|) < I_0(|z|)$, the estimate (27) follows. ■

Now, if one puts $\omega = \min(a, \beta)$, by means of (27) we get

$$\int_{\mathbb{R}} |K(x - \xi, t)| d\xi \leq e^{-at} + \sqrt{b} \pi t e^{-\omega t} \quad (28)$$

because $I_0(|z|) < \exp(|z|)$. As consequence, for all $t \leq T$, from (28) it follows that

$$\int_0^t d\tau \int_{\mathbb{R}} |K(x - \xi, t - \tau)| d\xi \leq (1 + \sqrt{b} \pi / \omega) T \quad (29)$$

while, for $t \rightarrow \infty$, the formula (27) implies

$$\begin{aligned} \int_0^t d\tau \int_{\mathbb{R}} |K(x - \xi, t)| d\xi &\leq \frac{1}{a} + \pi \sqrt{b} \int_0^\infty \tau e^{-\frac{\beta+a}{2} \tau} I_0\left(\frac{a-\beta}{2} \tau\right) d\tau = \\ &= \frac{1}{a} + \pi \sqrt{b} \frac{a + \beta}{2(a\beta)^{3/2}} = \beta_0. \end{aligned} \quad (30)$$

The integrals related to the kernel $K_1 = e^{-\beta t} * K$ defined by (21) can be estimated in the same way. As $|J_0| \leq 1$ and

$$\int_0^\infty E(\tau) d\tau = \frac{1}{a\beta} = \beta_1, \quad (31)$$

the following results can be stated:

Theorem 2.4 - For all $t \in (0, \infty)$, the kernels K, K_1 defined by (10), (21) are such that

$$\int_{\mathbb{R}} |K| d\xi \leq (1 + \sqrt{b} \pi t) e^{-\omega t}, \quad \int_{\mathbb{R}} |K_1| d\xi \leq E(t) \quad (32)$$

$$\int_0^t d\tau \int_{\mathbb{R}} |K| d\xi \leq \beta_0, \quad \int_0^t d\tau \int_{\mathbb{R}} |K_1| d\xi \leq \beta_1 \quad (33)$$

where $\omega = \min(a, \beta)$, $E(t)$ is given by (25) and the constants β_0, β_1 are defined in (30), (31). ■

At last we observe that

$$\int_{\mathbb{R}} K_1(x - \xi, t) d\xi = \chi(t), \quad \int_{\mathbb{R}} K(x - \xi, t) d\xi = (\partial_t + \beta) \chi(t) \quad (34)$$

with the function $\chi(t)$ defined by

$$\chi(t) = e^{-\frac{\beta+a}{2}t} \operatorname{sen}(\varrho t)/\varrho, \quad \varrho = \frac{1}{2} \sqrt{4b - (a - \beta)^2}. \quad (35)$$

The formulae (34), (35) can be verified by means of their \mathcal{L} transforms because the related Laplace integrals are endowed with absolute convergence.

3 Existence and uniqueness results

As for the data F and g of the problem \mathcal{P}_0 , we shall admit:

Assumption 3.1 *The function $g(x)$ is continuously differentiable and bounded together with $g'(x)$. The function $F(x, t, u)$ is defined and continuous on the set*

$$D \equiv \{ (x, t, u) : (x, t) \in \Omega_T, -\infty < u < \infty \} \quad (36)$$

and more is uniformly Lipschitz continuous in (x, t, u) for each compact subset of Ω_T . Moreover F is bounded for bounded u and there exists a constant C_F such that the estimate

$$|F(x, t, u_1) - F(x, t, u_2)| \leq C_F |u_1 - u_2| \quad (37)$$

holds for all (u_1, u_2) . ■

When the problem \mathcal{P}_0 admits a solution u , then u must satisfy the integral equation

$$u = \int_{\mathbb{R}} K(x - \xi, t) g(\xi) d\xi + \int_0^t d\tau \int_{\mathbb{R}} K(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi \quad (38)$$

because of the properties of K stated by theorem 2.2 and the assumption 3.1 on the data. On the other hand, let $u(x, t)$ be a solution of (38) which is continuous and bounded so that also $F(x, t, u(x, t))$ is continuous and bounded by assumption 3.1. Then one can verify that u satisfies (1) owing to the properties of K , F and g . So, it's possible to conclude that

Theorem 3.1 - *The differential problem (1) admits a unique solution only if the integral equation (38) has a unique solution which is continuous and bounded in Ω_T .* ■

Consider now the integral equation (38) and, for small time $0 \leq t \leq \theta$ (with $\theta < T$), let

$$\|v\|_\theta = \sup_{\substack{-\infty < x < \infty \\ 0 \leq t \leq \theta}} |v(x, t)| \quad (39)$$

and

$$\mathcal{B}_\theta \equiv \{v(x, t) : v \in C[(-\infty, \infty) \times [0, \theta]] \text{ and } \|v\|_\theta < \infty\} \quad (40)$$

Further, let denote by $f_1 * f_2$ the convolution

$$f_1(x, t) * f_2(x, t) = \int_{\mathbb{R}} f_1(\xi, t) f_2(x - \xi, t) d\xi \quad (41)$$

and by $\mathcal{F}v$ the mapping

$$\mathcal{F}v(x, t) = K(x, t) * g(x) + \int_0^t K(x, t - \tau) * F(x, \tau, v(x, \tau)) d\tau \quad (42)$$

The foregoing remarks imply that \mathcal{B}_θ is a Banach space and (42) maps \mathcal{B}_θ in \mathcal{B}_θ and represents a continuous map for $0 \leq t \leq \theta$. In fact, from (37) and (29) it results

$$\|\mathcal{F}v_1 - \mathcal{F}v_2\|_\theta \leq C_F \theta (1 + \pi\sqrt{b}/\omega) \|v_1 - v_2\|_\theta. \quad (43)$$

Hence, when one selects θ such that $C_F \theta (1 + \pi\sqrt{b}/\omega) < 1$ then \mathcal{F} is a contraction of \mathcal{B}_θ into \mathcal{B}_θ and so has a unique fixed point $u(x, t) \in \mathcal{B}_\theta$.

In order to show that this result holds also for $0 < t \leq T$, it suffices to proceed by induction and so the integral equation (38) has a unique solution for $0 < t \leq (n + 1)\theta$, whatever the positive integer n may be.

Theorem 3.2 - *When the data (F, g) satisfy the assumption 3.1, then the initial value problem \mathcal{P}_0 defined in (1) admits a unique regular solution $u(x, t)$ in Ω_T .* ■

4 A priori estimates and continuous dependence

According to the assumption 3.1, in the class of bounded solutions, let

$$\|g\| = \sup_{\mathbb{R}} |g(x)|, \quad \|u\|_T = \sup_{\Omega_T} |u(x, t)|, \quad \|F\| = \sup_D |F(x, t, u)|.$$

The solution $u(x, t)$ of the problem \mathcal{P}_0 depends continuously upon the data, owing to the following theorem.

Theorem 4.1 - *Let u_1, u_2 be solutions of the problem \mathcal{P}_0 related to the data (g_1, F_1) and (g_2, F_2) which satisfy the assumption 3.1. Then there exists a positive constant C such that*

$$\|u_1 - u_2\|_T \leq C \sup_{\mathcal{R}} |g_1 - g_2| + C \sup_D |F_1(x, t, u) - F_2(x, t, u)|,$$

where C depends on C_F, T and the parameters a, b, β .

Proof - The integral equations (38) for u_1 and u_2 are differenced and the estimates of theorem 2.4 allow to apply the Gronwall lemma. ■

Remark 4.1 - When the function F is a known linear source $f(x, t)$ then (38) gives the *explicit solution* of the linear problem \mathcal{P}_0

$$u = K * g + \int_0^t K(x, t - \tau) * f(x, \tau) d\tau. \quad (44)$$

By means of this formula and the basic properties of the kernel K stated in theorems 2.3 and 2.4, it's possible to estimate u and its derivatives together with their asymptotic behavior. ■

In the non linear case, the integral equation (38) implies a priori estimates. For example:

Theorem 4.2 - *When the data (g, F) of the non linear problem \mathcal{P}_0 verify the assumption 3.1, then it results:*

$$\|u(x, t)\|_T \leq \beta_0 \|F\| + \|g\| [e^{-at} + \sqrt{b} \pi t e^{-\omega t}] \quad (45)$$

where the constant β_0 is defined by (30) and depends on a, b, β .

Proof - It suffices to apply to (38) the estimates (28) and (33). ■

5 Excitable models and travelling pulses

As it is well known, a biological system is excitable if a stimulus of sufficient size can initiate a travelling pulse which will propagate through the medium. One of the most relevant examples of systems with excitable behavior is given by neural communications by nerve cells via electrical signalling [11, 15].

Let $u(x, t)$ a transmembrane potential and let $v(x, t)$ a variable associated with the contributions to the membrane current from sodium, potassium and other ions. Then a simple example of excitable models is the Fitzhugh Nagumo system (FHN) [21, 22]:

$$\begin{cases} \frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} - v + f(u) \\ \frac{\partial v}{\partial t} = bu - \beta v \end{cases} \quad (46)$$

where $\varepsilon > 0$ is the diffusion coefficient related to the axial current in the axon, while b and β are positive constants that characterize the model's kinetic. Further

$$f(u) = u(a - u)(u - 1) \quad (47)$$

In absence of diffusion ($\varepsilon = 0$), the system (46)-(47) is related to a two variable phase plane system with the phase portrait that varies according to different values of the parameters a, b, β . In fact, when $(1 - a)^2 < 4b/\beta$, there is only the stable steady state O with $u = 0, v = 0$. If $(1 - a)^2 > 4b/\beta$, there are two other steady state A and B with

$$u_A = \frac{1}{2} [a + 1 - \sqrt{(1 - a)^2 - 4b/\beta}] < a, \quad v_A = \frac{b}{\beta} u_A$$

and

$$u_B = \frac{1}{2} [a + 1 + \sqrt{(1 - a)^2 - 4b/\beta}] < 1, \quad v_B = \frac{b}{\beta} u_B.$$

The state B is as locally stable as O while the state A between them is unstable. So, a sufficiently strong perturbation to a point $(\bar{u}, 0)$ with $\bar{u} > u_A$ may induce a large phase trajectory excursion, while a small perturbation around O ($\bar{u} < u_A$) rapidly decays towards O . All this is typical of the threshold behavior.

Moreover, if $z = x - ct$, travelling wave solutions of (46) are functions $u(z), v(z)$ such that

$$\varepsilon \ddot{u} + c\dot{u} + f(u) - v = 0, \quad c\dot{v} + bu - \beta v = 0. \quad (48)$$

A solitary pulse is meant to be a solution of (48) with appropriate boundary conditions for $z \rightarrow \pm\infty$. A classical meaningful example [10,21,22] is related to the particular case of $b = 0$ and

$$\lim_{z \rightarrow -\infty} u(z) = 1, \quad \lim_{z \rightarrow \infty} u(z) = \lim_{z \rightarrow \infty} v(z) = 0. \quad (49)$$

In this case the exact analytical solution of (48), (49) is

$$u(z) = \frac{1}{1 + e^{\gamma z}}, \quad v(z) = 0 \quad (50)$$

where: $\gamma = 1/\sqrt{2\varepsilon}$, $c = \sqrt{\varepsilon/2} (1 - 2a)$. Several other solutions of the equation (46)₁ which behave like (50) are well known in literature [4,5,8, 18,23,29] and are all bounded solutions that verify the assumption 3.1. As consequence, the results stated in section 3 and 4 are meaningful for the analysis of the evolution of (FHN) model in all of the space.

Let

$$u(x, 0) = u_0, \quad v(x, 0) = v_0 \quad (x \in \mathbb{R}) \quad (51)$$

the initial conditions related to the system (46) and let

$$f(u) = -a u + \varphi(u) \quad \text{with} \quad \varphi = u^2 (a + 1 - u). \quad (52)$$

As (46)₂ implies

$$v = v_0 e^{-\beta t} + b \int_0^t e^{-\beta(t-\tau)} u(x, \tau) d\tau, \quad (53)$$

when one puts $F(x, t, u) = \varphi(u) - v_0(x) e^{-\beta t}$, then the initial value problem (46)-(51) can be given the form (1) to obtain u in terms of the data.

For this we apply the formula (38) with $g = u_0$ and put

$$K \otimes F = \int_0^t d\tau \int_{\mathbb{R}} K(x - \xi, t - \tau) F[\xi, \tau, u(\xi, \tau)] d\xi.$$

Because of (24) it is $K \otimes (v_0 e^{-\beta t}) = v_0 * K_1$, and so

$$K \otimes F = K \otimes \varphi - K_1 * v_0(x).$$

Therefore by (38) we get:

$$u(x, t) = u_0 * K - v_0 * K_1 + \varphi \otimes K \quad (54)$$

and this formula, together with (53), allows to obtain also $v(x, t)$ in terms of the data. If we observe that

$$\begin{aligned}
& \int_0^t e^{-\beta(t-\tau)} K_1(x, \tau) d\tau = K_2(x, t) = \\
& = \int_0^t \frac{e^{-\frac{x^2}{4\varepsilon y} - a y - \beta(t-y)}}{2\sqrt{\pi\varepsilon y}} \sqrt{\frac{t-y}{by}} J_1(2\sqrt{by(t-y)}) dy \quad (55)
\end{aligned}$$

by means of (53)-(55) we get

$$v(x, t) = v_0 e^{-\beta t} + b[u_0 * K_1 - v_0 * K_2 + \varphi \otimes K_1]. \quad (56)$$

When the non linear part $\varphi(u)$ of $f(u)$ is negligible, or can be approximated by a constant φ_0 , then (54)-(56) represent the explicit solution (u, v) of the linear case. Otherwise (54) is an integral equation for the unknown u that is identical to (38) and then (56) can be applied for v .

Because of theorems 3.2 and 4.1, the following result can be stated.

Theorem 5.1 - *When the data (u_0, v_0) satisfy the hypotheses 3.1, then the initial value problem related to (FHN) system (46) has a unique regular solution in the space of bounded solution. Moreover this solution depends continuously upon the data.* ■

In order to apply theorem 4.2, an estimate like (32) must be set also for the kernel K_2 . From (55) it results:

$$\int_{\mathbb{R}} |K_2(x - \xi, t)| d\xi \leq \int_0^t e^{-ay - \beta(t-y)} (t - y) dy \leq t E(t) \quad (57)$$

so that, if

$$\|u_0\| = \sup_{\mathbb{R}} |u_0(x)|, \quad \|v_0\| = \sup_{\mathbb{R}} |v_0(x)|, \quad \|\varphi\| = \sup_D |\varphi(u)|,$$

by means of (57) and theorem 2.4 the formulae (54) - (56) imply the following conclusion.

Theorem 5.2 - *For initial data (u_0, v_0) compatible with assumption 3.1, the solution of the problem (46) - (51) satisfies the estimates:*

$$\begin{cases} |u| \leq \|u_0\| (1 + \pi\sqrt{b}t) e^{-\omega t} + \|v_0\| E(t) + \beta_0 \|\varphi\| \\ |v| \leq \|v_0\| e^{-\beta t} + b(\|u_0\| + t\|v_0\|) E(t) + b\beta_1 \|\varphi\| \end{cases} \quad (58)$$

with $\beta_0, \beta_1, E(t)$ defined by (30), (31), (25). ■

The analysis and the stability of solutions of nonlinear binary reaction - diffusion systems of PDE's, as well as the existence of global compact attractors, have been discussed in a great number of recent and interesting papers (see e. g. [6,17,26,27,31]). As it is well known, the (FHN) system admits arbitrary large invariant rectangles Σ containing $(0,0)$ so that the solution (u,v) , for all times $t > 0$, lies in the interior of Σ when the initial data (u_o, v_o) belong to Σ . Besides, when both β and a are strictly positive, then the estimates (58) confirm that the large time behaviour of the solution is determined only by the reaction mechanism because the explicit terms depending on the initial data are exponentially vanishing.

Finally we observe that when $b = 0$, $\beta = 0$ there is no bounded set that attracts all solutions and the asymptotic analysis needs the use of characteristic parameters such as fast and slow times, together with rigorous estimates uniformly valid for all t . To this extent, the basic properties of the kernel K might be useful for a rigorous singular perturbation analysis [10,14,22].

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